Derivatives of Exponential and Logarithmic Functions

-e^x is particularly useful in modeling exponential growth.
-An interesting application as it applies to calculus is

$$\lim_{h\to 0}\frac{e^h-1}{h}=1 \qquad \text{“Ratio”}$$

-This creates an interesting relationship between the function e^x and its derivative:

$$\frac{d}{dx}\left(e^x\right)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$=\lim_{h\to 0}\frac{e^x\bullet e^h-e^x}{h} \qquad e^x\bullet e^h=e^{x+h}$$

$$=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}$$

$$=\lim_{h\to 0}e^x\bullet\frac{e^h-1}{h}$$

$$=e^x\bullet 1=e^x$$

-So the derivative of e^x is the function itself e^x!

$$\frac{d}{dx}e^x=e^x$$

-If u is a differentiable function of x

$$\frac{d}{dx}e^u=e^u\frac{du}{dx}$$

Example-How does the Flu spread?

-The spread of the flu is modeled by

$$P(t) = \frac{100}{1+e^{3-t}}$$

where P is the total number of students infected t days after the first case was noticed.

a) Estimate the initial number of students infected

$$P(0) = \frac{100}{1+e^{3-0}} = 5 \ (rounded)$$

b) How fast is the flu spreading after 3 days?

$$\frac{dP}{dt} = \left(100\left(1+e^{3-t}\right)^{-1}\right) = -100\left(1+e^{3-t}\right)^{-2} \bullet \frac{d}{dt}\left(1+e^{3-t}\right)$$

$$= -100\left(1+e^{3-t}\right)^{-2} \bullet \left(0+e^{3-t} \bullet \frac{d}{dt}(3-t)\right)$$

$$= -100\left(1+e^{3-t}\right)^{-2}\left(e^{3-t}(-1)\right)$$

$$= \frac{100e^{3-t}}{\left(1+e^{3-t}\right)^{2}}$$

Derivative of a^x

-What about an exponential function with a base other then e?

-We will assume the base is positive and no zero since $y = 1^x$ is constant.

So, for $a > 0$, and $a \neq 1$

$$\frac{d}{dx}\left(a^u\right) = a^u \bullet \ln a\left(\frac{du}{dx}\right)$$

Example

-At what point does $y = 2^t - 3$ have a tangent line with slope 21?

$$\frac{d}{dt}\left(2^t - 3\right) = 2^t \ln\left(2\right) - 0 = 2^t \ln\left(2\right)$$

$$2^t \ln\left(2\right) = 21$$

$$2^t = \frac{21}{\ln\left(2\right)}$$

$$\ln\left(2^t\right) = \ln\left(\frac{21}{\ln\left(2\right)}\right)$$

$$t - \ln\left(2\right) = \ln\left(21\right) - \ln\left(\ln\left(2\right)\right)$$

$$t = \frac{\ln\left(21\right) - \ln\left(\ln\left(2\right)\right)}{\ln\left(2\right)} \approx 4.921$$

$$y = 2^t \bullet 3 = 27.297$$

$$\left(4.9, 27.3\right)$$

Derivative of $\ln(x)$

-Since we know $\frac{d}{dx}e^x$ we can find the derivative of it's inverse $\ln(x)$

$$y = \ln(x)$$

$$e^y = x$$

$$\frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x)$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{dy}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}$$

Example

-A line with slope m passes through the origin and is tangent to the graph of $y = \ln(x)$. What is the value of m?

-We weren't given a point!!
-We know

-coordinate is $(a, \ln(a))$

-slope is $m = \frac{1}{a}$

-Since the tangent passes through the origin

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln(a)}{a}$$

-Set the two formulas for slope equal to one another

$$\frac{\ln(a)}{a} = \frac{1}{a}$$

$$\ln(a) = 1$$

$$e^{\ln a} = e^{1}$$

$$a = e$$

$$m = \frac{1}{e}$$

Derivative of $\log_a x$

-The derivative of $\log_a x$ comes from the change of base formula

$$\log_a x = \frac{\ln x}{\ln a}$$

-For $a > 0$ and $a \neq 1$

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}$$

Power Rule for Arbitrary Powers

-If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Example

$$y = x^{\sqrt{2}}$$

$$\frac{dy}{dx} = \sqrt{2}x^{\left(\sqrt{2}-1\right)}$$

Example

$$y = \left(2+\sin\left(3x\right)\right)^{\pi}$$

$$\frac{dy}{dx} = \pi\left(2+\sin\left(3x\right)\right)^{\pi-1}\left(\cos\left(3x\right)\right)\left(3\right)$$

$$= 3\pi\left(2+\sin\left(3x\right)\right)^{\pi-1}\left(\cos\left(3x\right)\right)$$

Example

Find $\frac{dy}{dx}$ for $y = x^x$

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{dy}{dx}\left(\ln y\right) = \frac{dy}{dx}\left(x \ln x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \bullet \ln\left(x\right) + x \bullet \frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\ln x + 1\right)$$

$$\frac{dy}{dx} = x^{x}\left(\ln x + 1\right)$$